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A Galerkin method for a class of steady, two-dimensional, incompressible, laminar boundary-layer flows

By CHEN-CHI HSU
Department of Applied Mechanics and Engineering Science,
University of Michigan, Ann Arbor

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A Galerkin method is proposed for a class of boundary-layer flow problems. In this method, the assumed solution is composed of an auxiliary function and a series solution. The representing functions used in the series solution are orthonormal eigenfunctions, closely related to that of the boundary-layer equation, and are independent of the initial condition, as well as the boundary conditions. The reduced system of stiff, first-order, nonlinear, ordinary differential equations then has diagonal dominance for the first part of the flow region. The proposed method has been tested on two representative flows. Numerical experiments show that highly accurate results can be obtained for the entire boundary-layer flow region, if the auxiliary function satisfying the initial and boundary conditions is chosen to satisfy the first compatibility condition of the upstream flow region. In fact the computation is rather simple, and the numerical integration of the reduced initial-value problem can be carried out up to separation with a fairly large step.

1. Introduction

The development of boundary-layer theory is closely related to the advancement in high performance of aeroplanes and ships, to the reduction of losses in turbomachines, and to providing a high level of space technology, Hence, in recent years there has been much attention directed towards developing a general technique capable of solving successfully and effectively the complicated set of boundary-layer equations. The techniques that come close to this desirable goal are the method of weighted residuals (Galerkin method), the difference–differential method (Hartree–Womersley method), and the implicit finite-difference method.

The implicit finite-difference techniques have been tested upon many boundary-layer flow problems with satisfactory results, by many researchers. The main difficulties confronting the general application of the method have been discussed by Flügge-Lotz (1967) and by Emmons (1970). The application of the Hartree–Womersley procedure to boundary-layer problems has been investigated in detail by Smith & Clutter (1963) and Clutter & Smith (1964). They reduced the governing partial differential equations to a system of second-order ordinary differential equations, by replacing the streamwise derivatives with finite-difference representations. Many flow examples had been tested; and the results obtained were generally accurate. However, in common with the implicit finite-difference method, there are some difficulties in application; special measures are therefore needed. For instance, a shooting method must be employed to solve a coupled system of two-point boundary-value problems which has a semi-infinite domain. Moreover, owing to the numerical stability condition, the smallest step size that can be taken in the streamwise direction is limited. This limitation is of particular relevance near separation, where a very small step size is generally required.

The Galerkin method and other methods of weighted residuals have the advantage over finite-difference methods that the operation required in a boundary-value problem can be carried out exactly. For this reason one can expect that solutions by a Galerkin procedure give better results with less computational effort. The Galerkin method has been applied to boundary-layer flow problems, with satisfactory results, by e.g. Dorodnitsyn (1962), Bethel (1966), Bossel (1970), MacDonald (1970), Mitra & Bossel (1971). One problem that occurs in the integration of the boundary-layer equations is the stiffness of the reduced system of first-order ordinary differential equations that one must ultimately solve. In a sense, all numerical approaches to boundary-layer problems can be regarded as different ways of overcoming the difficulty introduced by the stiffness of the system. The stiffness becomes more pronounced if the number of representing functions in a Galerkin method, or the number of grid points in a finite-difference procedure, is increased. It is a major obstacle in high-accuracy computations. Most of the realizations of Galerkin's procedure use only a very limited number of terms; and then the stiffness problem becomes less obvious. In general, boundary-layer profiles are fairly smooth, so that a representation with a limited number of terms is possible. This feature may well have accounted for the success of the approaches mentioned. We mention in passing that the stiffness problem is somewhat relieved if one introduces a co-ordinate transformation of the Falkner–Skan type. The rapid decrease of certain particular solutions, in representations using the original co-ordinates, can be attributed to the fact that the co-ordinate system does not follow the spreading of the boundary layer. Hence the eigensolutions for perturbations of a Falkner–Skan profile decrease much more slowly, because they express only the smoothing out of details of the initial profile, while the natural spreading of the profile is taken into account by the choice of the co-ordinate system.

The Galerkin method is not without deficiencies. The selection of a set of representing functions is crucial to the accuracy of the procedure. For instance, if the set of representing functions used is not complete, then the increasing order of approximation does not necessarily result in an improvement in accuracy. In most of the approaches mentioned, the representing functions used are either power series of the independent variable, or powers of some exponential functions. Of course, a continuous function can be approximated arbitrarily well by a polynomial of sufficiently high degree; but in numerical practice the degree of a polynomial one can handle in this manner is rather low. If one tries to approximate a profile by a polynomial of high degree, then extremely ill-behaved matrices will arise, and the method becomes intractable. This deficiency disappears if one uses orthogonal polynomials. A representation of this kind combined with some special method for integrating stiff systems of differential equations, such as that of Gear (1969), would make it possible to increase the number of terms to be carried.

In this study a complete set of orthonormal eigenfunctions is derived through a rational analysis and linearization of the governing boundary-layer equations, for a class of steady two-dimensional incompressible laminar flows. The eigenfunctions obtained have closed-form solutions, and can be predetermined and used in the Galerkin method for all conceivable nonlinear problems governed by the system of equations treated. In the proposed Galerkin procedure, the assumed solution is composed of two parts: an approximating function $F(\xi, \eta)$ and a series of representing functions. The auxiliary function $F(\xi, \eta)$ is to be selected to satisfy the inhomogeneous boundary condition at infinity, as well as to improve the convergence of the series solution. The boundary-layer problem is then reduced to an initial-value problem. The system of first-order nonlinear ordinary differential equations to be integrated has a diagonal-dominance form, at least in the region close to the initial station. Under these circumstances the solution of the system of stiff equations becomes rather simple (Guderley & Hsu 1972*a*).

Two representative boundary-layer flow problems have been chosen for testing the proposed Galerkin method: linearly retarded flow and flow past a circular cylinder. The computational method is rather simple and straightforward. The numerical experiments show that highly accurate results can be obtained for the entire boundary-layer flow region, if the auxiliary function $F(\xi, \eta)$ is chosen to satisfy the initial profile and the first compatibility condition of the upstream flow region.

2. Governing equations

For a steady, two-dimensional, incompressible, laminar flow past a submerged body the governing boundary-layer equations, made dimensionless by a characteristic length L and a characteristic velocity U_∞, are

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0, \tag{2.1}$$

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = U_1 \frac{dU_1}{dx_1} + \frac{1}{Re} \frac{\partial^2 u_1}{\partial y_1^2}. \tag{2.2}$$

u_1 and v_1 are the dimensionless velocity components in the x_1 and y_1 directions, respectively. $U_1(x_1)$ is the given external flow velocity; and $Re \equiv U_\infty L/\nu$ is the Reynolds number. The associated boundary conditions and the initial condition are assumed to be

$$u_1(x_1, 0) = 0, \quad v_1(x_1, 0) = 0, \tag{2.3}$$

$$u_1(x_1, y_1 \to \infty) \to U_1(x_1), \tag{2.4}$$

$$u_1(0, y_1) = u_i(y_1). \tag{2.5}$$

Equations (2.1)–(2.5) constitute the formulation of the class of boundary-layer flow problems to be studied. It is generally advantageous to transform the problem before a solution technique is employed.

Frequently, one carries out the transformations

$$x_2 = \int_0^{x_1} U_1(s)\,ds, \quad y_2 = Re^{\frac{1}{2}} U_1(x_1)\, y_1, \tag{2.6}$$

$$\left.\begin{aligned} U_1(x_1) = U_2(x_2), \quad u_1(x_1, y_1) = U_1(x_1)\, u_2(x_2, y_2), \\ v_1(x_1, y_1) = Re^{-\frac{1}{2}} U_1(x_1)\,[v_2(x_2, y_2) - y_2 u_2(x_2, y_2)\, d\ln U_2/dx_2]. \end{aligned}\right\} \tag{2.7}$$

These bring (2.1)–(2.5) into the standardized forms

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} = 0, \tag{2.8}$$

$$u_2 \frac{\partial u_2}{\partial x_2} + v_2 \frac{\partial u_2}{\partial y_2} = \frac{\partial^2 u_2}{\partial y_2^2} + (1 - u_2^2)\frac{d\ln U_2}{dx_2}, \tag{2.9}$$

$$u_2(x_2, 0) = 0, \quad v_2(x_2, 0) = 0, \tag{2.10}$$

$$u_2(x_2, y_2 \to \infty) \to 1, \tag{2.11}$$

$$u_2(0, y_2) = u_i(y_1(y_2))/U_1(0). \tag{2.12}$$

This system of transformed equations has the advantages over the original one that it does not depend on the Reynolds number, and that the boundary conditions are the same for all conceivable flow problems.

Next, one can reduce the number of dependent variables by one, by the von Mises transformation. Moreover, it is advantageous in numerical methods if a Falkner–Skan transformation is also incorporated so that the boundary-layer thickness, expressed in terms of the independent variables, will remain of the same order of magnitude through the entire flow region, or at least a major part of it. Accordingly, one introduces

$$\frac{\partial \Psi}{\partial y_2} = u_2(x_2, y_2), \quad \frac{\partial \Psi}{\partial x_2} = -v_2(x_2, y_2), \tag{2.13}$$

$$x = \int_0^{x_2} \frac{ds}{s}, \quad y = x_2^{-\frac{1}{4}} \Psi^{\frac{1}{2}}(x_2, y_2) \equiv x_2^{-\frac{1}{4}} \left[\int_0^{y_2} u_2(x_2, s)\,ds\right]^{\frac{1}{2}}, \tag{2.14}$$

$$u(x, y) = u_2^2(x_2, y_2), \quad U(x) = U_2(x_2). \tag{2.15}$$

Then one obtains, from (2.8)–(2.12),

$$\frac{\partial u}{\partial x} = \tfrac{1}{4} u^{\frac{1}{2}} \left[\frac{1}{y^2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3}\frac{\partial u}{\partial y}\right] + \tfrac{1}{4} y \frac{\partial u}{\partial y} + 2(1-u)\frac{d\ln U}{dx}, \tag{2.16}$$

$$u(x, 0) = 0, \quad u(x, y \to \infty) \to 1, \tag{2.17}$$

$$u(0, y) = u_0(y). \tag{2.18}$$

$u(x, y)$ has the dimensions velocity squared; and $u_0(y)$ is the known initial profile. For computational purposes, the lower integration limit in the first transformation of (2.14) must be replaced by a finite but small number ϵ. Hence one has

$$x = \ln\left(\frac{x_2}{\epsilon}\right) \quad \text{or} \quad x_2 = \epsilon e^x. \tag{2.19}$$

We also note from the second relation of (2.14) that the independent variable y is linearly proportional to either y_1 or y_2 in the vicinity of the submerged body. This fact is important in studying the behaviour of the function $u(x, y)$ in the region $y \ll 1$.

3. Initial profile $u_0(y)$

In principle, the initial profile can be arbitrarily prescribed. However, in order to obtain realistic examples we proceed in the following manner. For the class of boundary-layer flow problems considered, the external flow velocity $U_1(x_1)$ can be represented by a power series in x_1. Hence, for $x_1 \ll 1$, one has

$$U_1(x_1) \sim x_1^m, \tag{3.1}$$

where m is a constant parameter, depending on the flow problem. According to the transformations (2.6) and (2.14), one then finds

$$\frac{d \ln U}{dx} = \frac{m}{m+1} \equiv \tfrac{1}{2}\beta_0, \tag{3.2}$$

where β_0 as usual describes the wedge angle of a Falkner–Skan flow. Assuming that the initial profile $u_0(y)$ is given by a Falkner–Skan solution, one obtains, from (2.16) and (2.17),

$$\frac{d^2u_0}{dy^2} + \left[\frac{y^3}{u_0^{\frac{1}{2}}} - \frac{1}{y}\right]\frac{du_0}{dy} + 4\beta_0 \frac{y^2}{u_0^{\frac{1}{2}}}(1-u_0) = 0, \tag{3.3}$$

$$u_0(0) = 0, \quad u_0(y \to \infty) \to 1. \tag{3.4}$$

The familiar form of the Falkner–Skan equation, namely

$$f'''(\eta) + f(\eta)f''(\eta) + \beta_0[1 - f'^2(\eta)] = 0,$$

can be derived, from (3.3), by the relations

$$y^2 = f(\eta)/\sqrt{2}, \quad u_0(y) = [f'(\eta)]^2.$$

For sufficiently small values of y, the initial profile can be expanded as

$$u_0(y) = b_2 y^2 + b_3 y^3 + b_4 y^4 + \ldots. \tag{3.5}$$

The constant coefficients b_j are found to be

$$b_2 = \sqrt{2} f''(0), \quad b_3 = -\tfrac{4}{3}\beta_0 b_2^{-\frac{1}{2}}, \quad b_4 = -\tfrac{1}{3}\beta_0^2 b_2^{-2}, \quad \text{etc.} \tag{3.6}$$

The expansion (3.5) and its derivatives are needed in generating the initial profile by a direct integration of (3.3), since the coefficients of (3.3) are singular at $y = 0$.

For $y \gg 1$, the asymptotic behaviour of $u_0(y)$ can be found in the following manner. Let

$$u_0(y) = 1 + \epsilon(y), \tag{3.7}$$

where $\epsilon(y)$ is small. One then obtains from (3.3) the linearized equation

$$\frac{d^2\epsilon}{dy^2} + \left(y^3 - \frac{1}{y}\right)\frac{d\epsilon}{dy} - 4\beta_0 y^2 \epsilon = 0. \tag{3.8}$$

To recognize the relative importance of different terms as $y \to \infty$, we carry out the transformation

$$\epsilon(y) = y^{\frac{1}{2}} \exp\left(-\tfrac{1}{8}y^4\right)\bar{\epsilon}(y), \tag{3.9}$$

which removes the first derivative term. Equation (3.8) now takes the form

$$d^2\bar{\epsilon}/dy^2 - \left[\tfrac{1}{4}y^6 + (1+4\beta_0)\,y^2 + \tfrac{3}{4}y^{-2}\right]\bar{\epsilon} = 0. \tag{3.10}$$

To find an asymptotic solution, one first determines an expression of the form

$$\bar{\epsilon}(y) \sim y^{\mu} \exp\left(\zeta(y)\right). \tag{3.11}$$

$\zeta(y)$ is chosen so that (3.10) is satisfied in the highest power in y; and μ is chosen to suppress the term of next order induced by $\zeta(y)$. One obtains

$$\bar{\epsilon}(y) \sim y^{-\frac{3}{2}} \exp\left(\pm\tfrac{1}{8}y^4\right). \tag{3.12}$$

Substitution of (3.12) into (3.9) gives

$$\epsilon(y) \sim y^{-1} \quad \text{and} \quad \epsilon(y) \sim y^{-1}\exp\left(-\tfrac{1}{4}y^4\right). \tag{3.13}$$

The first relation in (3.13) is not suitable for problems of boundary-layer type considered. Therefore, an asymptotic solution for the initial profile is

$$u_0(y) = 1 - Ky^{-1}\exp\left(-\tfrac{1}{4}y^4\right). \tag{3.14}$$

K is an arbitrary constant, which can be determined by matching. The constant μ in (3.11) can also be chosen to satisfy the term of next order. One then obtains

$$\bar{\epsilon}(y) \sim y^{4\beta_0 - \frac{1}{2}} \exp\left(\tfrac{1}{8}y^4\right) \quad \text{and} \quad \bar{\epsilon}(y) \sim y^{-\frac{5}{2} - 4\beta_0}\exp\left(-\tfrac{1}{8}y^4\right).$$

Consequently one has another asymptotic solution

$$u_0(y) = 1 - K_1 y^{-2-4\beta_0}\exp\left(-\tfrac{1}{4}y^4\right). \tag{3.15}$$

In computation the asymptotic solution obtained, either (3.14) or (3.15) should be used for $y \gg 1$.

4. Representing functions

In order to find an orthonormal set of representing functions, which are closely related to the eigenfunctions of the operator occurring in the boundary-layer equation (2.16), and which are independent of the initial condition as well as boundary conditions, a further transformation must be carried out. First, one rewrites (2.16) in the form

$$\frac{\partial u}{\partial x} = \frac{u_0^{\frac{1}{2}}}{4}\left[\frac{1}{y^2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3}\frac{\partial u}{\partial y}\right] + \tfrac{1}{4}y\frac{\partial u}{\partial y} + 2(1-u)\frac{d\ln U}{dx} + \tfrac{1}{4}\left(u^{\frac{1}{2}} - u_0^{\frac{1}{2}}\right)\left[\frac{1}{y^2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3}\frac{\partial u}{\partial y}\right]. \tag{4.1}$$

This is based on the assumption that, for a certain interval in x, the profile $u(x, y)$ does not deviate much from the initial profile $u_0(y)$. Next, a transformation of the independent variable is carried out, to bring the predominant part of the operator into a standard form. Let

$$\xi = x, \quad \eta = \left[\frac{3}{2}\int_0^y s u_0^{-\frac{1}{4}}(s)\,ds\right]^{\frac{2}{3}}, \tag{4.2}$$

$$v(\xi, \eta) = u(x, y). \tag{4.3}$$

Equation (4.1) becomes

$$\frac{\partial v}{\partial \xi} = \frac{1}{4\eta}\left[\frac{\partial^2 v}{\partial \eta^2} + \left(y\eta\eta' - \frac{1}{2\eta} - \frac{1}{4\eta'}\frac{u_0'}{u_0}\right)\frac{\partial v}{\partial \eta}\right] + 2(1-v)\frac{d\ln U}{d\xi}$$
$$+\frac{1}{4\eta}\left[\left(\frac{v}{u_0}\right)^{\frac{1}{2}} - 1\right]\left[\frac{\partial^2 v}{\partial \eta^2} - \left(\frac{1}{2\eta} + \frac{1}{4\eta'}\frac{u_0'}{u_0}\right)\frac{\partial v}{\partial \eta}\right]; \quad (4.4)$$

and (2.17) and (2.18) give the conditions

$$v(\xi, 0) = 0, \quad v(\xi, \eta \to \infty) \to 1, \quad (4.5)$$

$$v(0, \eta) = v_0(\eta) \equiv u_0(y(\eta)). \quad (4.6)$$

The function $u_0(y)$ now appears only in the coefficient of the first derivative and in the nonlinear term. In order to separate out the dominant part of the coefficient, we study its behaviour at $\eta = 0$ and as $\eta \to \infty$. For $\eta \ll 1$, one obtains, from (3.5) and (4.2),

$$\eta = b_2^{-\frac{1}{6}} y\left[1 - \frac{1}{10}\frac{b_3}{b_2}y - \frac{1}{14}\left(\frac{b_4}{b_2} - 0{\cdot}59\frac{b_3^2}{b_2^2}\right)y^2 + O(y^3)\right]. \quad (4.7)$$

Then, by inversion of (4.7), one finds

$$y = b_2^{\frac{1}{6}}\eta\left[1 + \frac{1}{10}\frac{b_3}{b_2}b_2^{\frac{1}{6}}\eta + \frac{1}{14}\left(\frac{b_4}{b_2} - 0{\cdot}31\frac{b_3^2}{b_2^2}\right)b_2^{\frac{1}{3}}\eta^2 + O(\eta^3)\right]. \quad (4.8)$$

Hence, for $\eta \ll 1$, the coefficient of the first derivative in (4.4) is

$$\frac{1}{4\eta}\left[y\eta\eta' - \left(\frac{1}{2\eta} + \frac{1}{4\eta'}\frac{u_0'}{u_0}\right)\right] = \frac{1}{4\eta}\left[-\frac{1}{\eta} + O(\eta^0)\right]. \quad (4.9)$$

As for $\eta \gg 1$, one has

$$\eta = (\tfrac{3}{4}y^2)^{\frac{2}{3}}[1 + O(y^{-2})], \quad (4.10)$$

$$y = (\tfrac{4}{3})^{\frac{1}{2}}\eta^{\frac{3}{4}}[1 + O(\eta^{-\frac{3}{2}})], \quad (4.11)$$

$$\frac{1}{4\eta}\left[y\eta\eta' - \left(\frac{1}{2\eta} + \frac{1}{4\eta'}\frac{u_0'}{u_0}\right)\right] = \frac{1}{4\eta}[\tfrac{4}{3}\eta^2 + O(\eta^{\frac{1}{2}})]. \quad (4.12)$$

Displaying the dominant terms of the coefficient at $\eta = 0$ and $\eta \to \infty$, we write (4.4) in the form

$$\frac{\partial v}{\partial \xi} = \frac{1}{4\eta}\left[\frac{\partial^2 v}{\partial \eta^2} + \left(\tfrac{4}{3}\eta^2 - \frac{1}{\eta}\right)\frac{\partial v}{\partial \eta}\right] + Q(\eta)\frac{\partial v}{\partial \eta} + 2(1-v)\frac{d\ln U}{d\xi} + R(\xi, \eta), \quad (4.13)$$

where

$$Q(\eta) = \frac{1}{4\eta}\left[y\eta\eta' - \tfrac{4}{3}\eta^2 + \frac{1}{2\eta} - \frac{1}{4\eta'}\frac{u_0'}{u_0}\right], \quad (4.14)$$

$$R(\xi, \eta) = \frac{1}{4\eta}\left[\left(\frac{v}{u_0}\right)^{\frac{1}{2}} - 1\right]\left[\frac{\partial^2 v}{\partial \eta^2} - \left(\frac{1}{2\eta} + \frac{1}{4\eta'}\frac{u_0'}{u_0}\right)\frac{\partial v}{\partial \eta}\right]. \quad (4.15)$$

The set of representing functions, independent of the initial as well as the boundary conditions, can now be obtained from the eigensolutions of a problem in which only the first term of the right-hand side of (4.13) is considered. This gives an eigenvalue problem

$$\frac{d^2\gamma}{d\eta^2} + \left(\tfrac{4}{3}\eta^2 - \frac{1}{\eta}\right)\frac{d\gamma}{d\eta} + 4\lambda\eta\gamma = 0, \quad (4.16)$$

$$\gamma(0) = 0, \quad \gamma(\eta \to \infty) \to 0. \quad (4.17)$$

The eigensolutions of this problem can be obtained in closed form. The asymptotic behaviour of the solution of (4.16), for large η, suggests the hypothesis

$$\gamma(\eta) = \exp(-\tfrac{4}{9}\eta^3)\,\tilde{\gamma}(\eta). \tag{4.18}$$

This brings (4.16) into the form

$$\frac{d^2\tilde{\gamma}}{d\eta^2} - \left(\tfrac{4}{3}\eta^2 + \frac{1}{\eta}\right)\frac{d\tilde{\gamma}}{d\eta} + 4(\lambda - \tfrac{1}{3})\,\eta\tilde{\gamma} = 0. \tag{4.19}$$

This equation can then be solved by a series expansion. Eigensolutions satisfying the boundary conditions, (4.17), are obtained if the series solution terminates. Thus one finds

$$\lambda_k = k, \quad \gamma_k(\eta) = \eta^2 \exp(-\tfrac{4}{9}\eta^3)\,P_k(\eta), \tag{4.20}$$

for $k = 1, 2, \ldots$, where $P_k(\eta)$ is a polynomial of degree $(k-1)$ in η^3. To be specific,

$$P_k(\eta) = \sum_{m=0}^{k-1} a_{k,m}\eta^{3m}, \quad a_{k,m} = \frac{4(m-k)}{3m(3m+2)}\,a_{k,m-1}. \tag{4.21}$$

From (4.16) and (4.17), one finds the orthogonality conditions

$$\int_0^\infty \exp(\tfrac{4}{9}\eta^3)\,\gamma_j(\eta)\,\gamma_k(\eta)\,d\eta = 0 \quad \text{for} \quad j \neq k. \tag{4.22}$$

The norm for the eigenfunction $\gamma_j(\eta)$ can be computed in the following manner. Introduce the new variable

$$z = \tfrac{4}{9}\eta^3; \tag{4.23}$$

and write the polynomial $P_j(\eta)$ of (4.21) in the form

$$P_j(\eta(z)) = (\tfrac{9}{4})^{j-1}a_{j,j-1}L_{j-1}(z), \tag{4.24}$$

in which $L_j(z)$ is a Laguerre polynomial of degree j. Then the norm for the eigenfunction $\gamma_j(\eta)$ is given by

$$\tilde{N}(\gamma_j) = \tfrac{9}{8}(\tfrac{3}{2})^{\frac{1}{3}}[(\tfrac{9}{4})^{j-1}a_{j,j-1}]^2\int_0^\infty z^{\frac{2}{3}}e^{-z}[L_{j-1}(z)]^2\,dz. \tag{4.25}$$

The above integral can be expressed in terms of gamma functions (e.g. Krylov 1962). Hence, let

$$N(\gamma_j) \equiv \frac{\tilde{N}(\gamma_j)}{a_{j,0}^2} = \tfrac{9}{8}(\tfrac{3}{2})^{\frac{1}{3}}(\tfrac{9}{4})^{2(j-1)}(j-1)!\,\Gamma(j+\tfrac{2}{3})\left[\frac{a_{j,j-1}}{a_{j,0}}\right]^2. \tag{4.26}$$

Then the right-hand side of (4.26) can conveniently be computed using the second relation in (4.21) and the value $\Gamma(\tfrac{5}{3}) = 0{\cdot}90274531$. In fact, one has

$$N(\gamma_1) = 1{\cdot}16255859;$$

for $j \geqslant 2$, $N(\gamma_j)$ can be obtained by recurrence.

Accordingly, choose the first coefficient for the polynomial $P_k(\eta)$ of (4.21) so that

$$a_{k,0} = [N(\gamma_k)]^{-\frac{1}{2}}. \tag{4.27}$$

Then the eigenfunctions $\gamma_k(\eta)$ obtained are normalized; and one has the orthonormality conditions

$$\int_0^\infty \exp(\tfrac{4}{9}\eta^3)\,\gamma_j(\eta)\,\gamma_k(\eta)\,d\eta = \int_0^\infty \eta^4\exp(-\tfrac{4}{9}\eta^3)\,P_j(\eta)\,P_k(\eta)\,d\eta = \delta_{jk}. \tag{4.28}$$

(δ_{jk} is the Kronecker delta.)

In Guderley & Hsu (1972*b*) a procedure which is more closely patterned to the asymptotic theory of second-order differential equation was suggested. The essential point is a transformation of the dependent variable such that the first derivative term is eliminated. The corresponding comparison eigensolutions obtained are also proportional to the polynomial of (4.21). Both approaches were tested in a heat-transfer problem (Guderley & Hsu 1973) and in the preliminary study of a linearly retarded boundary-layer flow problem (Hsu 1973). The results indicate that for the application of the Galerkin method with a limited number of terms the transformation of the dependent variable is impractical. It generates more complicated expressions without an improvement in accuracy. The transformation would be necessary if one wanted to obtain a true asymptotic expression for large eigenvalues.

5. Galerkin's procedure

The transformed boundary-layer equation for $v(\xi,\eta)$ is now solved by the Galerkin method using for the representing functions the orthonormal eigenfunctions obtained in §4. Accordingly we write the assumed solution in the form

$$v(\xi,\eta) = F(\xi,\eta) + \sum_{k=1}^{\infty} C_k(\xi)\,\gamma_k(\eta), \tag{5.1}$$

where the function $F(\xi,\eta)$ is to be selected to satisfy the boundary condition at infinity, as well as to improve the convergence of the series solution. Substituting (5.1) into the governing equation (4.13), and eliminating the second derivative of $\gamma_k(\eta)$ by (4.16), one has

$$\sum_{k=1}^{\infty} \gamma_k(\eta)\left[\frac{dC_k}{d\xi}+\lambda_k C_k\right] = \sum_{k=1}^{\infty}\left[-2\gamma_k\frac{d\ln U}{d\xi}C_k + Q(\eta)\frac{d\gamma_k}{d\eta}C_k\right] + R(\xi,\eta)$$
$$+\frac{1}{4\eta}\left[F_{\eta\eta}+\left(y\eta\eta'-\frac{1}{2\eta}-\frac{1}{4\eta'}\frac{u_0'}{u_0}\right)F_\eta\right] + 2(1-F)\frac{d\ln U}{d\xi} - F_\xi. \tag{5.2}$$

Now, multiplying this equation by the weight function $\exp(\frac{4}{9}\eta^3)\,\gamma_j(\eta)$, and integrating the resulting equation from zero to infinity, then applying the orthonormality conditions (4.28), one is led to the following system of first-order nonlinear ordinary differential equations:

$$\frac{dC_j}{d\xi}+\lambda_j C_j = -2\frac{d\ln U}{d\xi}C_j + \sum_{k=1}^{\infty} A_{jk}C_k + B_j(\xi) + \frac{d\ln U}{d\xi}D_j(\xi)$$
$$+E_j(\xi)+H_j(\xi) \quad \text{for} \quad j = 1,2,\ldots, \tag{5.3}$$

where
$$A_{jk} = \int_0^\infty Q(\eta)\exp(\tfrac{4}{9}\eta^3)\,\gamma_j(\eta)\frac{d\gamma_k}{d\eta}\,d\eta, \tag{5.4}$$

$$B_j(\xi) = \frac{1}{4}\int_0^\infty \frac{1}{\eta}\left[F_{\eta\eta}+\left(y\eta\eta'-\frac{1}{2\eta}-\frac{1}{4\eta'}\frac{u_0'}{u_0}\right)F_\eta\right]\exp(\tfrac{4}{9}\eta^3)\,\gamma_j(\eta)\,d\eta, \tag{5.5}$$

$$D_j(\xi) = 2\int_0^\infty [1-F(\xi,\eta)]\exp(\tfrac{4}{9}\eta^3)\,\gamma_j(\eta)\,d\eta, \tag{5.6}$$

$$E_j(\xi) = \int_0^\infty R(\xi,\eta)\exp\left(\tfrac{4}{9}\eta^3\right)\gamma_j(\eta)\,d\eta, \tag{5.7}$$

$$H_j(\xi) = -\int_0^\infty F_\xi(\xi,\eta)\exp\left(\tfrac{4}{9}\eta^3\right)\gamma_j(\eta)\,d\eta. \tag{5.8}$$

Similarly, if one multiplies the assumed solution by the weight function then integrates it from zero to infinity, one obtains the initial conditions

$$C_j(0) = \int_0^\infty [v_0(\eta) - F(0,\eta)]\exp\left(\tfrac{4}{9}\eta^3\right)\gamma_j(\eta)\,d\eta. \tag{5.9}$$

This equation suggests that it is convenient to choose $F(\xi,\eta)$ so that

$$F(0,\eta) = v_0(\eta).$$

The stiffness of the system (5.3) expresses itself by the values of λ_j, $2d\ln U/d\xi$, the elements of matrix A_{jk}, and implicitly in the nonlinear term $E_j(\xi)$. The preliminary results obtained for Falkner–Skan flows have shown that A_{jk} is generally less than 15% of the λ_j (Hsu 1973). Hence, the linear part of the system (5.3) indeed has a diagonal-dominance form; and the integration method of Guderley & Hsu (1972*a*) for stiff equations is advantageous in solving the initial-value problem (5.3) and (5.9). In the present procedure, the values of $\lambda_j = j$ are not overly large, thus a finite system of (5.3) is only moderately stiff. This results from the introduction of the Falkner–Skan transformation, which partially anticipates the effect of boundary-layer thickening.

6. Selection of $F(\xi,\eta)$

In the proposed Galerkin method, the assumed solution for $v(\xi,\eta)$ consists of an approximating function $F(\xi,\eta)$ and a series solution in eigenfunctions $\gamma_k(\eta)$, (5.1). Since the eigenfunctions that satisfy the homogeneous boundary conditions at zero and at infinity are predetermined, the function $F(\xi,\eta)$ must be chosen to satisfy the boundary conditions of the flow problem

$$F(\xi,0) = 0, \quad F_\eta(\xi,0) = 0, \quad F(\xi,\eta\to\infty)\to 1. \tag{6.1}$$

Moreover, it is desirable and possible to take certain steps that improve the convergence of the series solution. This makes it guaranteed to work with a limited number of terms of the series solution in (5.1).

The selection of $F(\xi,\eta)$ is, therefore, based on the following observation. For $\eta \ll 1$, the representing functions $\gamma_k(\eta)$ in (5.1) have the power series development

$$\gamma_k(\eta) = \alpha_{2k}\eta^2 + \alpha_{5k}\eta^5 + \alpha_{8k}\eta^8 + \dots. \tag{6.2}$$

Hence, for $\eta = 0$ all the derivatives of order $3m$ and $3m+1$, $m = 1, 2, \dots$, are zero. On the other hand, these derivatives of the physical profile $v(\xi,\eta)$ generally do not vanish. Therefore, we have a situation similar to that which occurs in representing a sectionally continuous function by a Fourier series. An approximation in the sense of minimizing of square error does exist; but the series converges rather slowly as $\Sigma 1/n$. However, if the function and its first k derivatives are continuous, then the Fourier series for the function will converge more rapidly than the series $\Sigma\ 1/n^{k+1}$ (see e.g. Kantorovich & Krylov 1964). In the present problem, something similar would happen. The coefficients in the development of

$$[v(\xi,\eta)-F(\xi,\eta)]_{\eta\eta\eta}$$

would decrease rather slowly if $(v-F)_{\eta\eta\eta}$ were not zero at the boundary $\eta = 0$. The same, but to a lesser degree, would also happen if the fourth derivative of $(v-F)$ with respect to η were not zero at $\eta = 0$. It follows that the convergence of the series development in (5.1) can be improved if one chooses the function $F(\xi,\eta)$ so that the third and the fourth derivatives of $(v-F)$ with respect to η are zero at $\eta = 0$. Of course, one could also consider the sixth and the seventh derivatives of $(v-F)$; but their effect on the convergence of the series solution in (5.1) is much less pronounced.

The derivatives needed for these corrective terms are actually determined by the governing differential equation, the boundary conditions, and the second, the fifth, etc., derivative of $v(\xi,\eta)$ with respect to η. The relations so obtained are called the compatibility conditions. It is true that if one were to solve the governing partial differential equation perfectly, then the compatibility conditions would automatically be satisfied. Strictly speaking, the information contained in these conditions is not necessary for the formulation of the problem. However, in the present approach these conditions are important, for they allow one to obtain good approximations with a limited number of representing functions. We believe that the success of the Kármán–Pohlhausen method, and of certain versions of the Galerkin method, can be attributed to the fact that they take some of the compatibility conditions into account.

To find the relation for these derivatives, one substitutes the series development

$$u(x,y) = a_2(x)\,y^2+a_3(x)\,y^3+a_4(x)\,y^4+\dots \tag{6.3}$$

into (2.16), then compares the coefficients of terms that are of the same power in y. One finds

$$a_3(x) = -\tfrac{8}{3}d\ln U/dx\,a_2^{-\frac{1}{2}}(x), \tag{6.4}$$

$$a_4(x) = -\frac{3}{16}\frac{a_3^2(x)}{a_2(x)} = -\frac{4}{3}\left(\frac{d\ln U}{dx}\right)^2 a_2^{-2}(x). \tag{6.5}$$

These two compatibility conditions are indeed expressed in terms of the second derivative of $v(\xi,\eta)$, and the given outer velocity. For higher derivatives, one should differentiate (2.16) with respect to y. Then one can obtain the compatibility conditions $a_6(x)$ and $a_7(x)$ in terms of $a_5(x)$ and $a_2(x)$, and so on. To find the corresponding relations in terms of the variables ξ and η, one assumes that the initial profile for $y \ll 1$ is given by (3.5). Then, from the relations (4.7) and (4.8), one finds that

$$v_0(\eta) = b_2^{\frac{4}{3}}\eta^2+\tfrac{6}{5}b_2^{\frac{1}{2}}b_3\eta^3+b_2^{\frac{2}{3}}(\tfrac{8}{7}b_4+\tfrac{93}{350}b_3^2/b_2)\,\eta^4+\dots, \tag{6.6}$$

$$\begin{aligned} v(\xi,\eta) = {} & b_2^{\frac{1}{3}}\,a_2(\xi)\,\eta^2+b_2^{\frac{1}{2}}\left[a_3(\xi)+0{\cdot}2\frac{b_3}{b_2}a_2(\xi)\right]\eta^3 \\ & +b_2^{\frac{2}{3}}\left[a_4(\xi)+\frac{1}{7}\frac{b_4}{b_2}a_2(\xi)+\frac{b_3}{b_2}\left(\tfrac{3}{10}a_3-\frac{6}{175}\frac{b_3}{b_2}a_2\right)\right]\eta^4+\dots, \end{aligned} \tag{6.7}$$

where $a_j(\xi) = a_j(x)$ since $\xi = x$. Now the expressions for $v_{\eta\eta\eta}(\xi, 0)$ and $v_{\eta\eta\eta\eta}(\xi, 0)$ extracted from (6.7) are the counterparts of the compatibility conditions (6.4) and (6.5).

A function $F(\xi, \eta)$ by which the initial condition and the first two compatibility conditions can be satisfied is

$$F(\xi, \eta) = v_0(\eta) + X_1(\xi) f_1(\eta) + X_2(\xi) f_2(\eta), \tag{6.8}$$

where $v_0(\eta)$ is the initial profile, and

$$f_1(\eta) = \tfrac{1}{6}\eta^3 \exp(-\tfrac{4}{9}\eta^3), \quad f_2(\eta) = \tfrac{1}{24}\eta^4 \exp(-\tfrac{4}{9}\eta^3). \tag{6.9}$$

One notes that these functions satisfy the boundary conditions

$$f_1(0) = f_1'(0) = f_1''(0) = f_1^{\mathrm{iv}}(0) = f_1(\infty) = 0, \quad f_1'''(0) = 1, \tag{6.10}$$

$$f_2(0) = f_2'(0) = f_2''(0) = f_2'''(0) = f_2(\infty) = 0, \quad f_2^{\mathrm{iv}}(0) = 1. \tag{6.11}$$

Thus the approximating function given by (6.8) indeed satisfies the boundary conditions (6.1). The substitution of (6.8) into (5.1) gives the assumed solution in the form

$$v(\xi, \eta) = v_0(\eta) + X_1(\xi) f_1(\eta) + X_2(\xi) f_2(\eta) + \sum_{k=1}^{\infty} C_k(\xi)\, \gamma_k(\eta). \tag{6.12}$$

If one differentiates this equation twice with respect to η, then evaluates it at $\eta = 0$, one finds

$$a_2(\xi) = b_2 + b_2^{-\frac{1}{3}} \sum_{k=1}^{\infty} C_k(\xi)\, P_k(0). \tag{6.13}$$

Similarly, the third and the fourth derivatives of (6.12) give the explicit expression for $X_1(\xi)$ and $X_2(\xi)$, respectively. Assuming the validity of (3.6), and using (6.4) and (6.5), one obtains

$$X_1(\xi = x) = -16 \frac{d \ln U}{dx} \left(\frac{b_2}{a_2}\right)^{\frac{1}{3}} - \tfrac{8}{5}\beta_0 \frac{a_2}{b_2} + \tfrac{48}{5}\beta_0, \tag{6.14}$$

$$X_2(\xi = x) = 24 b_2^{-\frac{4}{3}} \left[-\frac{4}{3}\left(\frac{d \ln U}{dx}\frac{b_2}{a_2}\right)^2 + \tfrac{16}{15}\beta_0 \frac{d \ln U}{dx}\left(\frac{b_2}{a_2}\right)^{\frac{1}{3}} - \tfrac{1}{175}\beta_0^2 \left(19\frac{a_2}{b_2} + 16\right)\right]. \tag{6.15}$$

These equations indicate that the approximating function $F(\xi, \eta)$ must depend on the unknown function $a_2(\xi)$, if the compatibility conditions are to be satisfied exactly.

In principle, the best procedure for achieving highly accurate results would probably be obtained if one were to extrapolate for $X_1(\xi)$ and $X_2(\xi)$ the unknown function $a_2(\xi)$ of (6.13), through the interval of integration for which the computation is carried out to determine the coefficients $C_k(\xi)$, since only a limited number of terms in series is used in practice. This procedure would certainly require a substantial increase in computing effort; and, as indicated by the numerical experiment, special measures are needed in the vicinity of separation. One must remember that both $X_1(\xi)$ and $X_2(\xi)$ are introduced for the sole purpose of improving the convergence of the series solution in (6.12). In general, a considerable amount of improvement can be achieved even if the $X_1(\xi)$ and $X_2(\xi)$ used in computation are only approximations to their exact expressions, (6.14) and (6.15). A systematic search for the approximations for $X_1(\xi)$ and $X_2(\xi)$ is carried out in numerical examples. It shows that a simple approximation to the unknown function $a_2(\xi)$ can be effectively used in (6.14) and (6.15).

7. Evaluation of integrals

The numerical integration of the initial-value problem (5.3) and (5.9) requires the evaluation of integrals (5.4)–(5.8). Using the relations (4.20) and (6.8), and introducing the new independent variable (4.23), i.e.

$$z = \tfrac{4}{9}\eta^3, \tag{7.1}$$

the integrals (5.4)–(5.8) become

$$A_{jk} = \frac{3}{4}\int_0^\infty e^{-z}\eta Q(\eta)\,[2-\tfrac{4}{3}\eta^3+\eta P_k'/P_k]\,P_j(\eta)\,P_k(\eta)\,dz, \tag{7.2}$$

$$B_j(\xi) = \frac{3}{4}\sum_{m=0}^{2} X_m(\xi)\int_0^\infty e^{-z}\frac{1}{4\eta}\left[f_m''+f_m'\left(y\eta\eta'-\frac{1}{2\eta}-\frac{1}{4}\frac{v_0'}{v_0}\right)\right]e^z P_j(\eta)\,dz, \tag{7.3}$$

$$D_j(\xi) = \frac{3}{2}\int_0^\infty e^{-z}(1-v_0)\,e^zP_j(\eta)\,dz - \frac{3}{2}\sum_{m=1}^{2} X_m(\xi)\int_0^\infty e^{-z}f_m(\eta)\,e^zP_j(\eta)\,dz, \tag{7.4}$$

$$H_j(\xi) = -\frac{3}{4}\sum_{m=1}^{2}\frac{dX_m}{d\xi}\int_0^\infty e^{-z}f_m(\eta)\,e^z P_j(\eta)\,dz, \tag{7.5}$$

$$\begin{aligned} E_j(\xi) = \frac{3}{4}\sum_{m=0}^{2} X_m(\xi)\int_0^\infty e^{-z}G(\xi,\eta)\,S_m(\eta)\,P_j(\eta)\,dz \\ +\frac{3}{4}\sum_{k=1}^{\infty} C_k(\xi)\int_0^\infty e^{-z}\,G(\xi,\eta)\,T_k(\eta)\,P_j(\eta)\,dz, \end{aligned} \tag{7.6}$$

where $$X_0(\xi) \equiv 1, \quad f_0(\eta) \equiv v_0(\eta),$$

and $$G(\xi,\eta) \equiv \left[\sum_{m=0}^{2} X_m(\xi)\frac{f_m(\eta)}{v_0(\eta)} + \eta^2\exp(-\tfrac{4}{9}\eta^3)\sum_{k=1}^{\infty} C_k(\xi)\frac{P_k(\eta)}{v_0(\eta)}\right]^{\frac{1}{2}} - 1, \tag{7.7}$$

$$S_m(\eta) \equiv \frac{1}{4\eta}\left[f_m''-\left(\frac{1}{2\eta}+\frac{1}{4}\frac{v_0'}{v_0}\right)f_m'\right]\exp(\tfrac{4}{9}\eta^3), \tag{7.8}$$

$$T_k(\eta) \equiv \frac{1}{4\eta}\left[\left(\frac{1}{2\eta}-\tfrac{4}{3}\eta^2-\frac{1}{4}\frac{v_0'}{v_0}\right)\left(2\eta-\tfrac{4}{3}\eta^4+\eta^2\frac{P_k'}{P_k}\right)-4\lambda_k\eta^3\right]P_k(\eta). \tag{7.9}$$

One notes that all the integrands involved are known, except $G(\xi,\eta)$ of (7.7), which depends on the solution $C_k(\xi)$.

The integrals in (7.2)–(7.6) now have the form that can be evaluated by the Gaussian–Laguerre quadrature formula

$$\int_0^\infty e^{-z}f(z)\,dz \simeq \sum_{m=1}^{M} A_m f(z_m). \tag{7.10}$$

The set of values for A_m and the roots of the Laguerre polynomial z_m for different M have been tabulated by Stroud & Secrest (1966). If the integrand $f(z)$ is a polynomial of degree not greater than $2M-1$, then the quadrature formula (7.10) gives the exact value for the integral. For the present problem, the integrals in (7.2)–(7.5) can be evaluated beforehand, but the integrals resulting from the nonlinear term (7.6) must be evaluated at each integration step twice, if a predictor–corrector method is employed to solve the initial-value problem (5.3) and (5.9). In applications, the choice of M is determined by the desire to obtain accurate results, as well as to keep the computing effort as small as possible. For the class of problems considered, if one uses N representing functions in the assumed solution (6.12), then, according to our experience, $M > \frac{3}{2}N$ gives sufficient accuracy.

8. Remarks about the integration method

The numerical integration method employed in the computation deals with a system of first-order ordinary differential equations in the form

$$d\mathbf{y}/dx + \Lambda \mathbf{y} = \mathbf{r}(x, \mathbf{y}),$$

in which Λ is a diagonal matrix whose elements may be very large (Guderley & Hsu 1972*a*). The important feature of the method is that the effect of Λ is taken into account analytically, while the right-hand side $\mathbf{r}(x, \mathbf{y})$ is approximated by a polynomial of degree n in x. It is then possible to carry out the integration procedure by a predictor–corrector scheme. If the matrix Λ is zero, or if the term $\Lambda\mathbf{y}$ is incorporated in the right-hand side, the method becomes an ordinary predictor–corrector method. On the other hand, if the right-hand side is a known function of x only, then one obtains the analytic solution for $\mathbf{y}$, and the method is numerically stable for any step size. Therefore, one can expect to have stability for a fairly large integration step, if the stiffness of the right-hand side is small compared with that of $\Lambda\mathbf{y}$. In the actual computation, the right-hand side is approximated by a polynomial of degree four. This approximation is tested by comparing the difference between the predicted and the corrected values of $\mathbf{y}$ with the allowable truncation error. This check can then be used to control the step size of the integration. All the numerical results obtained in this study are based on the accuracy of 10^{-6} in both the initiation phase and the prediction–correction phase.

9. Linearly retarded flow

The boundary-layer flow problem with a linearly retarded mainstream flow velocity was introduced by von Kármán & Millikan (1934), to study boundary-layer separation. Later, the problem was solved by Howarth (1938), by a series-expansion method. Since then it has been considered one of the standard problems to test newly proposed computational methods. This problem is selected here as the first example to test the proposed Galerkin method; for the problem has been solved by many different methods, and accurate results are available for comparison. We assume that the dimensionless external velocity has the form

$$U_1(x_1) = 1 - \tfrac{1}{8}x_1. \tag{9.1}$$

This indicates that the initial profile can be given by the Blasius solution. One recalls that a different expression for $U_1(x_1)$ can be obtained by a simple transformation of the independent variable.

In the proposed Galerkin method, the final system of equations to be integrated is expressed with respect to the variable ξ. Therefore, $U_1(x_1)$ must be expressed as a function of ξ. Following the transformations (2.6) and (2.14) or (2.19), one finds

$$x_1 = 8[1-(1-\tfrac{1}{4}\epsilon e^{\xi})^{\frac{1}{2}}]. \tag{9.2}$$

Hence (9.1) becomes

$$U(\xi) = (1-\tfrac{1}{4}\epsilon e^{\xi})^{\frac{1}{2}}. \tag{9.3}$$

In the computations, we use $\epsilon = 10^{-6}$. Moreover we use a limited number of representing functions N in the assumed solution, i.e.

$$v(\xi,\eta) = v_0(\eta)+X_1(\xi)f_1(\eta)+X_2(\xi)f_2(\eta)+\sum_{k=1}^{N} C_k(\xi)\,\gamma_k(\eta); \tag{9.4}$$

and we define the dimensionless wall shear stress as

$$\tau_0(x_1) \equiv \frac{Re^{\frac{1}{2}}}{\rho U_\infty^2}\,\tau(\xi,0) = \frac{U^2(\xi)}{2(\epsilon e^{\xi})^{\frac{1}{2}}}\,a_2(\xi). \tag{9.5}$$

The latter can be computed from (6.13), (9.2) and (9.3).

As the first numerical experiment for the proposed method, we choose the approximating function $F(\xi,\eta) = v_0(\eta)$, i.e. $X_1(\xi) = X_2(\xi) = 0$. This implies that the assumed solution (9.4) satisfies the compatibility conditions of the initial profile only. The wall shear stress then computed with $N = 10$, as well as those of Howarth, are given in figure 1. As expected, the results are acceptable only in a small region $0 < x_1 < 0{\cdot}1$. Additional computation with $N = 20$ has resulted in a very slight improvement; the results are accurate in $0 < x_1 < 0{\cdot}2$. In the next trial experiment, we consider also the first correction term $X_1(\xi)$, (6.14), in which the unknown function $a_2(\xi)$ is roughly approximated by the first term of (6.13), b_2. The wall shear stress, then computed with $N = 10$, is now acceptable in a larger region $0 < x_1 < 0{\cdot}6$ (figure 1). These facts confirm the analysis that the accuracy of the approximation (9.4) depends rather strongly on how it satisfies the compatibility conditions. In fact, good approximations can be obtained, even if (9.4) satisfies the first compatibility condition only approximately.

The previous experiment showed that better results can be obtained for the entire flow region if $a_2(\xi)$ is chosen more properly in (6.14). One notes from (4.13) or (5.3) that the solution depends upon the known function $d\ln U/d\xi$. It is, therefore, legitimate to assume in the approximation for $X_1(\xi)$ that $a_2(\xi)$ is a function of $d\ln U/d\xi$. It probably suffices to approximate (6.13) by the simple linear relation

$$a_2(\xi) = a_2(0)+K_2\left[\frac{d\ln U}{d\xi}-\left.\frac{d\ln U}{d\xi}\right|_{\xi=0}\right], \tag{9.6}$$

and use it in the approximation for $X_1(\xi)$ and $X_2(\xi)$. For the present flow example, the first half of the flow region can in fact be described approximately by

$$a_2(\xi) = b_2+\frac{b_2-0{\cdot}2677}{0{\cdot}1262}\,\frac{d\ln U}{d\xi}. \tag{9.7}$$

With the approximation (9.7) for $X_1(\xi)$, and $N = 10$, the wall shear stress computed as a function of x_1 is given in figures 1 and 2. It shows that the method generally gives good results for the entire flow region. An additional run with

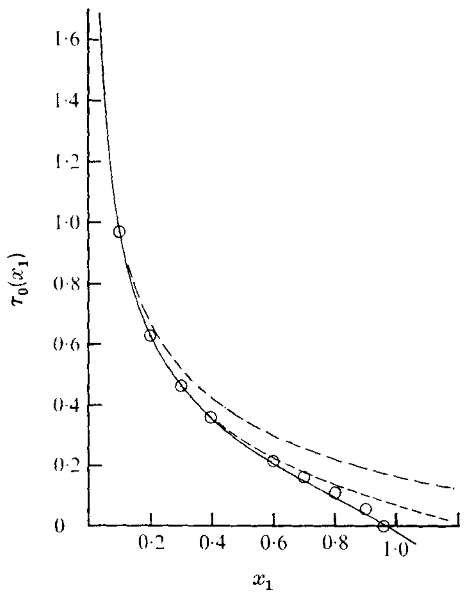

FIGURE 1

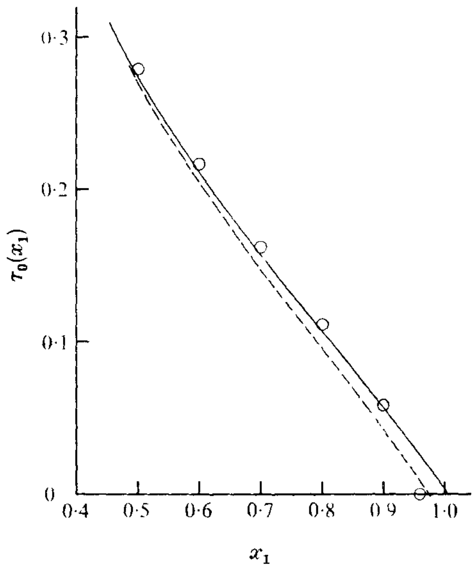

FIGURE 2

FIGURE 1. Wall shear stress for linearly retarded flow, for $X_0(\xi) = 0$ and $N = 10$ in (9.4): —·—, $X_1(\xi) = 0$; – – – –, $a_2(\xi) = b_2$; ——, $a_2(\xi)$ given by (9.7); ○, Howarth's results.

FIGURE 2. Wall shear stress for linearly retarded flow, for $X_2(\xi) = 0$ and $a_2(\xi)$ given by (9.7): – – –, $N = 10$; ——, $N = 20$; ○, Howarth's results.

$N = 20$ has been carried out; and the results obtained are also presented in figure 2. The figure indicates clearly that extremely accurate results for most of the boundary-layer flow region can be obtained by the present approach. However, in the near vicinity of the separation point, the results obtained are rather poor. This deficiency could be attributed to the fact that the representing functions are derived on the basis of flow without separation, and that the approximation (9.7) for $a_2(\xi)$ is not appropriate in that region. Thus, it would require a large number of representing functions to obtain accurate results in the vicinity of the separation point. We mention in passing that the singular behaviour at separation seems to be relaxed somewhat in the present approach. Consequently, the integration process for solving (5.3) can be carried up to the separation point with fairly large step size.

Additional numerical experiments have been carried out with the full assumed solution (9.4), in which $a_2(\xi)$ in (6.14) and (6.15) is approximated by a relation similar to (9.7). The results obtained for $\tau_0(x_1)$ are slightly better than those given in figure 2, up to $x_1 = 0{\cdot}8$; but they deteriorate further downstream. That the results obtained in the vicinity of separation are poorer is because $X_2(\xi)$, given by (6.15), depends rather strongly on $a_2(\xi)$, and the approximation used for $a_2(\xi)$ is not appropriate in that region.

The integration of (5.3) confirms that its linear operator has a diagonal-dominance form, since the step size is increased from 0·1 to 0·4, which is about eight times (for $N = 20$) larger than the maximum step size allowed by the stability of the ordinary predictor–corrector method. As the nonlinear effect becomes important downstream, the step size is automatically decreased, and $\Delta\xi = 0{\cdot}013$ near the separation. The computer time required for each run with $N = 10$ is about 1 min of CDC 6400, of which about 15 s are spent on generating the initial profile; for the runs with $N = 20$, it is about 150 s. The velocity profiles computed from (9.4) showed that the Falkner–Skan transformation, introduced in (2.14), is quite appropriate; for the boundary-layer thickness of the velocity, equal to 0·999, has increased only slightly in the entire flow region, from $\eta = 2{\cdot}41$ at the initial station to $\eta = 2{\cdot}50$ near separation.

Finally, the Galerkin procedure has been carried out exactly, in the sense that the unknown function $a_2(\xi)$ is computed from (6.13), at each integration step, for $X_1(\xi)$ and $X_2(\xi)$. As one would have expected, the results obtained are better than those of figure 2, for most of the flow region $0 < x_1 < 0{\cdot}85$. But poorer results are obtained in the vicinity of separation. This indicates that a large number of terms would be needed in (9.4) to obtain accurate results in that region. Moreover, the computation time required is drastically increased, owing to a sizeable reduction in step size in the vicinity of separation (e.g. in the run with $N = 20$, $\Delta\xi = 0{\cdot}013$ at $x_1 = 0{\cdot}6$; $0{\cdot}003$ at $x_1 = 0{\cdot}8$; $0{\cdot}0002$ at $x_1 = 0{\cdot}9$). Therefore, we can conclude that the present approach is not practical for obtaining the solution for the entire flow region.

10. Flow over a circular cylinder

The boundary-layer flow past a circular cylinder is different from the linearly retarded flow : in it, there is a region of accelerating mainstream followed by one of decelerating mainstream. So this problem is particularly testing for the proposed Galerkin method. Here, the inviscid solution of the problem is considered as the known external velocity. Accordingly, one has

$$U_1(x_1) = 2\sin x_1. \tag{10.1}$$

Hence, the Faulkner–Skan solution of a stagnant flow can be used as the initial profile. Analogously to the first example, one obtains, from (2.6), (2.19) and (10.1),

$$\cos x_1 = 1 - \tfrac{1}{2}\epsilon e^{\xi}, \tag{10.2}$$

$$U(\xi) = [\epsilon e^{\xi}(4-\epsilon e^{\xi})]^{\frac{1}{2}}. \tag{10.3}$$

Again the value $\epsilon = 10^{-6}$ is used in computations. Based on the assumed solution (9.4), the wall shear stress for the problem is defined as

$$\tau_0(x_1) \equiv \frac{Re^{\frac{1}{2}}}{\sqrt{8}\,\rho U_\infty^2}\,\tau(\xi,0) = \frac{U^2(\xi)}{4(2\epsilon e^{\xi})^{\frac{1}{2}}}\,a_2(\xi). \tag{10.4}$$

This can be computed from (6.13), (10.2) and (10.3).

In order to demonstrate again the significance of the compatibility conditions of the assumed solution (9.4), we first set $X_1(\xi) = X_2(\xi) = 0$. As one would expect, the computed wall shear stress ($N = 20$) is very accurate only for a short distance downstream of the initial station $0 < x_1 < 0{\cdot}5$ (figure 3). The results are considerably improved when $X_1(\xi)$ is included in the assumed solution, in which

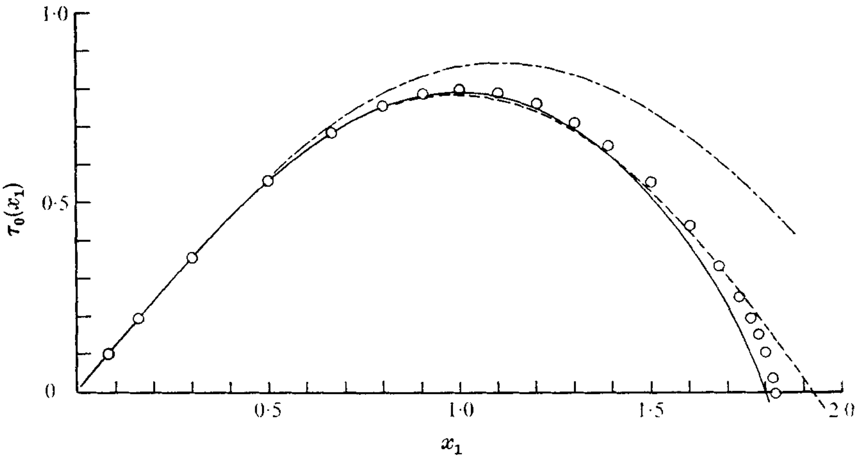

FIGURE 3. Wall shear stress for flow past a circular cylinder, for $X_2(\xi) = 0$ in (9.4): —·—, $X_1(\xi) = 0$, $N = 20$; - - -, $a_2(\xi) = b_2$, $N = 10$; ——, $a_2(\xi)$ given by (10.5), $N = 10$; ○, Terrill's numerical results.

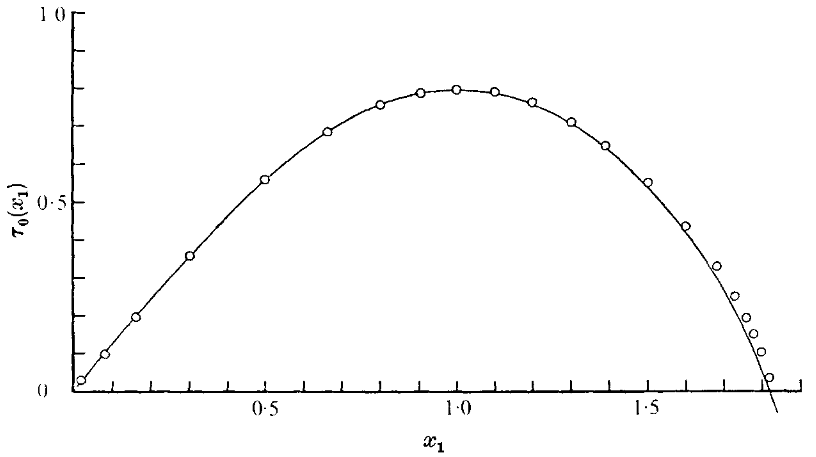

FIGURE 4. Wall shear stress for flow past a circular cylinder, for $X_2(\xi) = 0$, $a_2(\xi)$ given by (10.5) and $N = 20$: ○, Terrill's numerical results.

$a_2(\xi)$ is roughly approximated by the constant b_2. Figure 3 shows that $\tau_0(x_1)$, obtained with $N = 10$, is surprisingly accurate for most of the flow region. Again, poorer results are obtained for the vicinity of separation, and the separation point is given to an accuracy of within 6% of Terrill's (1960) numerical result.

Better results for the vicinity of separation can be obtained if the unknown function $a_2(\xi)$ for $X_1(\xi)$ is more properly approximated. For the present problem, the computational results also show that $a_2(\xi)$ is practically linearly proportional to $d\ln U/d\xi$ in the upstream flow region. Hence, it suffices to approximate $a_2(\xi)$ by (9.6). To be specific, we use

$$a_2(\xi) = \frac{0{\cdot}5020 - 0{\cdot}0539 b_2}{0{\cdot}4416} + \frac{b_2 - 1{\cdot}004}{0{\cdot}4416}\frac{d\ln U}{d\xi} \tag{10.5}$$

for $X_1(\xi)$, and set $X_2(\xi) = 0$ in (9.4). The results obtained with $N = 10$ are given in figure 3. It clearly indicates that $\tau_0(x_1)$ now behaves more like the numerical results. Consequently, one can expect more accurate results, particularly in the vicinity of separation, if more terms of the representing functions are considered. To confirm this expectation, we next use $N = 20$; the results obtained are given in figure 4. It is clear now that the present approach can give extremely accurate results for the entire flow region, if N is large enough. Again the integration process for (5.3) can be carried out up to the separation point, with fairly large step size. The computer time required for the run with $N = 20$ is about 3 min of CDC 6400; for $N = 10$, it is about 1 min. The velocity profile computed at various values of x_1 confirms that the boundary-layer thickness, measured with respect to the transformed variable η, remains the same order of magnitude throughout the entire flow region. For a dimensionless velocity equal to 0·999, η varies from 2·05, at the initial station, to 2·19 at separation.

Additional computations have been carried out for other numerical experiments treated in the linearly retarded flow problem. The results give the same information as those cited in §2.

11. Concluding remarks

For the class of boundary-layer flow problems considered, the several transformations carried out in the analysis prove to be beneficial, since the computational effort of the proposed Galerkin method is rather simple and straightforward. One need provide only the initial profile and the external flow velocity for the reduced initial-value problem, and the integration process can then be carried out directly from the initial station to the point of separation with fairly large step size. Furthermore, the method provides a qualitative assurance on the accuracy and the reliability of the results computed, by checking how the coefficient of the series solution $C_k(\xi)$ behaves as k increases. For instance, poorer results in the vicinity of separation are reflected by the fact that $C_k(\xi)$ either decreases very slowly, or starts to increase as k increases.

We conclude, from numerical experiments on the two representative problems, that the complete set of representing functions found (which is independent of the initial and boundary conditions) can indeed be used for all conceivable problems of the class considered, and that the method can effectively give highly accurate results for the entire boundary-layer flow region, if the auxiliary function $F(\xi, \eta)$ of (6.8) is chosen to satisfy only the first compatibility condition of the upstream flow region. The accuracy of the result, of course, depends on the number of representing functions used in computation. In fact, decent results for the entire flow region can be obtained with a reasonable amount of computer time (about 3 min of CDC 6400) if one uses 20 representing functions.

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